Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021–8928(04)00052-8

J. Appl. Maths Mechs, Vol. 68, No. 3, pp. 391–397, 2004

0021–8928/$—see front matter

THE STRUCTURE OF THREE-DIMENSIONAL PERIODIC BOUNDARY LAYERS IN A CONTINUOUSLY STRATIFIED FLUID†

M. A. DAVYDOVA and Yu. D. CHASHECHKIN

Moscow

e-mail: lisean@rol.ru; chakin@ipmnet.ru

(*Received* 15 *January* 2003)

Small three-dimensional motions of a slightly viscous stratified fluid, generated by vertical and torsional oscillations of part of the surface of an infinite vertical cylinder of arbitrary cross-section, are investigated. The asymptotic method of boundary functions is used to analyse the structure of periodic motions. It is shown that two types of boundary layers are formed, one of which possesses the properties of the Stokes boundary layer in homogeneous fluid, while the other one, namely, the internal wave boundary layer, is a specific feature of heterogeneous media, whose thickness depends on both the wave frequency and the buoyancy frequency. On changing to the case of a homogeneous fluid, the viscous and internal boundary layers merge. © 2004 Elsevier Ltd. All rights reserved.

During the past few years considerable attention has been devoted to analysing the boundary layers that are produced both on the bounding surfaces and on the free surface of a perturbed viscous fluid [1]. Consideration of the boundary effects enables one to construct exact solutions of the problem of generating two-dimensional and three-dimensional internal waves in a linear [2] and non-linear [3] formulation, which considerably extends the number of scenarios of the non-linear mechanisms by which internal waves are formed and evolve [4]. In this connection it is of interest to carry out a more detailed calculation of the parameters of the wave layers on a periodically moving surface in a continuously stratified fluid. Although this problem had been investigated by Stokes in the case of a homogeneous viscous fluid, a similar analysis for three-dimensional periodic motions in a continuously stratified fluid has not been carried out until now.

The system of equations that describes the periodic motions of a continuously stratified slightly viscous media relates to a class of singularly disturbed equations, for solving which a number of asymptotic and numerical methods have been developed. In the present paper the structure of periodic motions is analysed by the asymptotic method of boundary functions [5]. The solution is constructed in the form of an asymptotic expansion in a small parameter, which possesses the properties of a degenerate system inside the domain obeys the boundary conditions due to the introduction, into the asymptotic form, of boundary functions that decay exponentially with distance from the boundary. The viscosity is chosen as the small parameter of the asymptotic expansion. This technically simple method of analysing singularly perturbed problems has been successfully used to investigate fluid oscillations, the propagation of sound and other problems of fluid dynamics [6–8].

1. FORMULATION OF THE PROBLEM

The linearized system of equations in the Boussinesq approximation, which describes small three-dimensional motions of an incompressible viscous stratified fluid with an impurity in a gravitational field, has the form

$$\begin{aligned} &\rho_0\frac{\partial \mathbf{u}}{\partial t} = -\operatorname{grad}p + \rho_0\nu\Delta\mathbf{u} - \rho g\mathbf{e}_z \\ &\frac{\partial \rho}{\partial t} + u_3\frac{d\rho_0}{dz} = 0, \quad \operatorname{div}\mathbf{u} = 0 \end{aligned} \qquad (1.1)$$

†*Prikl. Mat. Mekh.* Vol. 68, No. 3, pp. 437–444, 2004.

where $\mathbf{u} = (u_1, u_2, u_3)$ is the velocity vector, p and ρ are the dynamic pressure the density, $\rho_0(z) = \rho_{00}\exp(-z/\Lambda)$ is the unperturbed density, Λ is the buoyancy scale, g is the acceleration due to gravity, ν is the kinematic viscosity and $\mathbf{e}_z$ is the unit vector directed along the Z axis.

The motion of the fluid occurs due to harmonic oscillations with a frequency ω of the part of a vertical cylinder of arbitrary cross-section, whose axis coincides with the direction of the gravity field, where the boundary conditions on the cylinder surface have the form

$$\mathbf{un}|_\Gamma = 0, \quad \mathbf{u\tau}_1|_\Gamma = U^1(x, y, z)e^{-i\omega t}, \quad \mathbf{u\tau}_2|_\Gamma = U^2(x, y, z)e^{-i\omega t} \tag{1.2}$$

where $\mathbf{n}$ is the outward normal to the fairly smooth surface of the cylinder Γ, τ_1 and τ_2 are the unit vectors oriented at each point of the surface Γ along the principal orthogonal directions, and, $U^1(x, y, z)$ and $U^2(x, y, z)$ are given functions.

Introducing dimensionless coordinates, time and velocity as the ratio of the corresponding physical quantities to their constant values characteristic for the given problem, and assuming that the dimensionless viscosity ν is equal to Re^{-1}, from system (1.1) we obtain the following dimensionless system (for convenience all the notation is preserved)

$$\begin{aligned} -i\omega\rho_0\mathbf{u} &= -\operatorname{grad}p + \nu\rho_0\Delta\mathbf{u} - \rho g\mathbf{e}_z \\ -i\omega\rho + u_3\frac{d\rho_0}{dz} &= 0, \quad \operatorname{div}\mathbf{u} = 0 \end{aligned} \tag{1.3}$$

To reduce the algebraic manipulations, we introduce into our analysis a representation of the velocity, with the help of which the amplitude of the velocity is defined as follows:

$$\mathbf{u} = \nabla\times(\mathbf{e}_z\Psi) + \nabla\times\nabla\times(\mathbf{e}_z\Phi) \tag{1.4}$$

Using representation (1.4), from system (1.3) we obtain the governing system of the problem for the function Ψ and Φ

$$[\omega^2\Delta - i\omega\nu\Delta^2 - N^2\Delta_2]\Phi = 0 \quad (\omega - i\nu\Delta)\Psi = 0; \quad \Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{1.5}$$

where N^2 is the square of the dimensionless buoyance frequency.

Taking (1.4) into account, from conditions (1.2) we obtain the boundary conditions for system (1.5) whose implicit form will be presented below.

2. OSCILLATIONS OF THE CYLINDER IN A MEDIUM WITH LOW VISCOSITY

If ν is a small dimensionless parameter: $\nu = \varepsilon^2$, $0 < \varepsilon \leqslant 1$, from system (1.5) we obtain

$$[\omega^2\Delta - i\omega\varepsilon^2\Delta^2 - N^2\Delta_2]\Phi = 0, \quad (\omega - i\varepsilon^2\Delta)\Psi = 0 \tag{2.1}$$

System (2.1) is singularly perturbed, since the differential operators occurring in it contain small parameters at higher-order derivatives.

On changing to an ideal fluid ($\varepsilon = 0$) we obtain the degenerate system

$$L_0\Phi^r \equiv \left[\frac{\partial^2}{\partial z^2} + \left(1 - \frac{N^2}{\omega^2}\right)\Delta_2\right]\Phi^r = 0, \quad \omega\Psi^r = 0 \tag{2.2}$$

The operator L_0 depends on the parameter ω, and we therefore consider two cases: if $|\omega| > N$, the operator L_0 is of elliptic type, whereas, if $|\omega| < N$, this operator will be a hyperbolic operator. From the physical point of view these two cases are distinguished by the possibility of the existence of stable waves in the part of space far from the cylinder boundary.

To describe the solution near the boundary, we introduce a local system of coordinates (r, σ_1, σ_2), where r is the distance from the point $M(r, \sigma_1, \sigma_2)$ to the boundary Γ along the normal M_0M to $\Gamma(M_0 \in \Gamma)$, and σ_1, σ_2 are the curvilinear coordinates of the point M_0 and Γ. If the vicinity

$$\Gamma_\delta = (0 \le r \le \delta) \times (0 < \sigma_1 \le \Sigma_1) \times (-\infty < \sigma_2 \le +\infty)$$

is sufficiently small (that is, the parameter δ is sufficiently small), a one-to-one correspondence exists between the coordinates (x, y, z) and (r, σ_1, σ_2).

From condition (1.2) and taking into account representation (1.4), we obtain the boundary conditions for system (2.1)

$$\left[\frac{1}{H_2}\frac{\partial\Psi}{\partial\sigma_1} + \frac{\partial^2\Phi}{\partial\sigma_2\partial r}\right]_\Gamma = 0, \quad \left[\frac{1}{H_2}\frac{\partial^2\Phi}{\partial\sigma_2\partial\sigma_1} - \frac{\partial\Psi}{\partial r}\right]_\Gamma = u^1(\sigma_1, \sigma_2)$$
$$\frac{1}{H_2}\left[\frac{\partial H_2}{\partial r}\frac{\partial\Phi}{\partial r} + H_2\frac{\partial^2\Phi}{\partial r^2} + \frac{\partial}{\partial\sigma_1}\left(\frac{1}{H_2}\frac{\partial\Phi}{\partial\sigma_1}\right)\right]_\Gamma = -u^2(\sigma_1, \sigma_2) \tag{2.3}$$

where H_2 is the Lamé parameter ($H_1 = H_3 = 1$), and 0, $u^1(\sigma_1, \sigma_2)$ and $u^2(\sigma_1, \sigma_2)$ are the components of the vector of the velocity of motion of the boundary in the local system of coordinates. The functions $u^1(\sigma_1, \sigma_2)$ and $u^2(\sigma_1, \sigma_2)$, defined on the cylinder surface, are fairly smooth and finite with respect to the variable σ_2, and it is assumed that

$$\lim_{\sigma_2 \to \infty} \int_{-\infty}^{\sigma_2} u'_{\sigma_1}(\sigma_1, \eta)d\eta = 0$$

Relations (2.3) specify the values of the projections of the vector of the velocity of motion of the boundary on to the axis of local system of coordinates.

Following the procedure employed in [5], the asymptotic form of the solution of problem (2.1) with boundary conditions (2.3) will be sought in the form

$$\Phi(x, y, z, \varepsilon) = \Phi^r(x, y, z, \varepsilon) + \Pi\Phi(\rho, \sigma_1, \sigma_2, \varepsilon) =$$
$$= \sum_{i=0}^{\infty} \varepsilon^i \Phi_i^r(x, y, z) + \sum_{i=0}^{\infty} \varepsilon^i \Pi_i\Phi(\rho, \sigma_1, \sigma_2) \tag{2.4}$$
$$\Psi(x, y, z, \varepsilon) = \Pi\Psi(\rho, \sigma_1, \sigma_2, \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \Pi_i\Psi(\rho, \sigma_1, \sigma_2); \quad \rho = \frac{r}{\varepsilon}$$

where $\Phi^r(x, y, z, \varepsilon)$ is the regular expansion, which describes the solution far from the boundary and $\Pi\Psi(\rho, \sigma_1, \sigma_2, \varepsilon)$ and $\Pi\Phi(\rho, \sigma_1, \sigma_2, \varepsilon)$ are the boundary layer corrections, which make a considerable contribution to the solution near the boundary.

Substituting expansions (2.4) into system (2.1) we separate the equations for the regular and boundary layer terms, in the equations for the boundary functions we change to new variables, make the extension $r = \varepsilon\rho$, and then expand the coefficients of the equations in powers of ε. To find the terms of the boundary layer expansions, we have the system

$$\left[-\frac{1}{\varepsilon^2}\left(i\omega\frac{\partial^4}{\partial\rho^4} - (\omega^2 - N^2)\frac{\partial^2}{\partial\rho^2}\right) - \frac{b(0)}{\varepsilon}\left(2i\omega\frac{\partial^3}{\partial\rho^3} - (\omega^2 - N^2)\frac{\partial}{\partial\rho}\right) + \sum_{l=0}^{\infty} \varepsilon^l M_l\right]\Pi\Phi = 0 \tag{2.5}$$

$$\left[\omega - i\sum_{l=0}^{\infty} \varepsilon^l N_l\right]\Pi\Psi = 0 \tag{2.6}$$

where M_l and N_l are linear differential operators of order no higher than the second and $b(0)$ is the zeroth term in the expansion of coefficient $b = H_2^{-1}\partial H_2/\partial r$ in powers of ε.

In a similar manner the boundary conditions are obtained from conditions (2.3)

$$\left[a(0)\frac{\partial\Pi\Psi}{\partial\sigma_1} + \frac{\partial^2\Phi^r}{\partial\sigma_2\partial n} + \frac{1}{\varepsilon}\frac{\partial^2\Pi\Phi}{\partial\sigma_2\partial\rho}\right]_\Gamma = 0 \tag{2.7}$$

$$\left[a(0)\frac{\partial^2(\Phi^r + \Pi\Phi)}{\partial\sigma_2\partial\sigma_1} - \frac{1}{\varepsilon}\frac{\partial\Pi\Psi}{\partial\rho}\right]_\Gamma = u^1(\sigma_1, \sigma_2) \tag{2.8}$$

$$\left[b(0)\frac{\partial\Phi^r}{\partial n} + \frac{1}{\varepsilon}b(0)\frac{\partial\Pi\Phi}{\partial\rho} + \frac{\partial^2\Phi^r}{\partial n^2} + \frac{1}{\varepsilon^2}\frac{\partial^2\Pi\Phi}{\partial\rho^2} + \right.$$
$$\left. + a(0)d_1(0)\frac{\partial(\Phi^r + \Pi\Phi)}{\partial\sigma_1} + a^2(0)\frac{\partial^2(\Phi^r + \Pi\Phi)}{\partial\sigma_1^2}\right]_\Gamma = -u^2(\sigma_1, \sigma_2) \tag{2.9}$$

where $a(0)$ and $d_1(0)$ are the zeroth terms in the expansions of the coefficients $a = H_2^{-1}$, $d_1 = \partial H_2^{-1}/\partial\sigma_1$ in powers of ε.

In addition, we require that all Π-functions tend to zero as $\rho \to \infty$:

$$\Pi(\rho, \sigma_1, \sigma_2) \to 0 \quad \text{or} \quad \rho \to \infty \tag{2.10}$$

Equating terms of like powers of ε in relations (2.5)–(2.10), we find in the zeroth and first approximations

$$\Pi_0\Phi = 0, \quad \Pi_1\Phi = 0, \quad \Pi_0\Psi = 0$$

The boundary layer function $\Pi_1\Psi$ is determined from relations (2.6), (2.8) and (2.10)

$$\omega\Pi_1\Psi - i\frac{\partial^2\Pi_1\Psi}{\partial\rho^2} = 0$$

$$\left.\frac{\partial\Pi_1\Psi}{\partial\rho}\right|_{\rho=0} = -u^1(\sigma_1, \sigma_2), \quad \Pi_1\Psi(\rho \to \infty) \to 0$$

From this we obtain

$$\Pi_1\Psi(\rho, \sigma_1, \sigma_2) = -\frac{u^1(\sigma_1, \sigma_2)}{\lambda_1}e^{\lambda_1\rho}, \quad \lambda_1 = \sqrt{\frac{\omega}{2}}(i-1) \tag{2.11}$$

In the next approximation we arrive at the system

$$i\omega\frac{\partial^4}{\partial\rho^4}\Pi_2\Phi - (\omega^2 - N^2)\frac{\partial^2}{\partial\rho^2}\Pi_2\Phi = 0$$

$$\left.\frac{\partial^2\Pi_2\Phi}{\partial\rho^2}\right|_{\rho=0} = -u^2(\sigma_1, \sigma_2), \quad \Pi_2\Phi(\rho \to \infty) \to 0$$

which has the solution

$$\Pi_2\Phi(\rho, \sigma_1, \sigma_2) = -\frac{u^2(\sigma_1, \sigma_2)}{\lambda_2^2}e^{\lambda_2\rho}, \quad \lambda_2 = \sqrt{\frac{\omega^2 - N^2}{2\omega}}(i-1) \tag{2.12}$$

The terms of the regular expansion $\Phi_l^r (l = 0, 1, 2)$ are defined from the problems of the form

$$\left[\frac{\partial^2}{\partial z^2} + \left(1 - \frac{N^2}{\omega^2}\right)\Delta_2\right]\Phi_l^r = 0, \quad \left.\frac{\partial^2\Phi_l^r}{\partial z\partial n}\right|_\Gamma = f_l(\sigma_1, \sigma_2) \tag{2.13}$$

When $|\omega| < N$ the solution of problem (2.13) that vanishes at infinity can be constructed using the additional condition

$$\lim_{\sigma_2 \to \infty}\int_{-\infty}^{\sigma_2} f_l(\sigma_1, \eta)d\eta = 0 \tag{2.14}$$

Assuming henceforth that the surface Γ is a Lyapunov surface, we will seek the solution of problem (2.13) that decays at infinity in the form of a Fourier integral

$$\Phi_l^r = \int_{-\infty}^{+\infty} \oint_\gamma \mu_{lk}(P) G_k(M, P) e^{ikz} dl_P dk \tag{2.15}$$

where γ is the line of intersection of the surface Γ and the plane z = const, and $G_k(M, P)$ is Green's function of the second boundary-value problem for Helmholtz's equation outside a circle of a certain radius a_k that lies wholly inside the contour γ. The function $G_k(M, P)$ exists and can be represented explicitly as

$$G_k(M, P) = \frac{1}{2\pi}\Bigg[H_0^{(1)}(c_k R_{MP}) - \frac{J_0'(c_k a_k)}{H_0^{(1)'}(c_k a_k)} H_0^{(1)}(c_k r_0) H_0^{(1)}(c_k r) -$$

$$-2\sum_{n=1}^{\infty} \frac{J_n'(c_k a_k)}{H_n^{(1)'}(c_k a_k)} H_n^{(1)}(c_k r_0) H_n^{(1)}(c_k r) \cos n(\varphi - \varphi_0)\Bigg]$$

where

$$R_{MP} = \sqrt{(x-\xi)^2 + (y-\zeta)^2}, \quad P(\xi, \zeta) \in \gamma, \quad c_k = \frac{|k|\omega}{\sqrt{N^2 - \omega^2}}$$

$H_n^{(1)}(c_k r)$ is the Hankel function of the first kind, $J_n(c_k r)$ is the Bessel function, and (r, φ), (r_0, φ_0) are coordinates of the points M and P in a polar system of coordinates. The potential density $\mu_{lk}(P)$ obeys Fredholm's equation of the second kind

$$\pi\mu_{lk}(P_0) - \oint_\gamma \mu_{lk}(P) \frac{\partial G_k(P, P_0)}{\partial n_{P_0}^{in}} dl_P = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{\sigma_2} f_l(\sigma_{1P_0}, \eta) e^{-ik\sigma_2} d\eta\, d\sigma_2, \quad P_0 \in \gamma \tag{2.16}$$

σ_{1P_0} is the coordinate of the point P_0 on the contour γ and $\partial/\partial n_{P_0}^{in}$ is the derivative along the inward normal to γ at the point P_0.

Since the kernel of integral operator, which occurs in Eq. (2.16), is weakly polar one and the corresponding homogeneous equation has only a trivial solution for a special choice a_k, Eq. (2.16) is uniquely solvable.

In the case when $|\omega| > N$ the solution of problem (2.13), which decays at infinity, is sought in the form

$$\Phi_l^r = \int_{-\infty}^{+\infty} \oint_\gamma \mu_{lk}(P) K_0(c_k R_{MP}) e^{ikz} dl_P dk, \quad c_k = \frac{|k|\omega}{\sqrt{\omega^2 - N^2}}$$

where $K_0(c_k R_{MP})$ is the McDonald function. The density $\mu_{lk}(P)$ obeys an equation that differs from (2.16) by substituting $K_0(c_k R_{PP_0})$ for $G_k(P, P_0)$. Further investigations are carried out in a similar way.

Note that to extract the unique physical solution of problem (2.13) in the case when $|\omega| > N$, it is sufficient to pose the condition for the solution to decay at infinity (unlikely the case when $|\omega| < N$, when the principle of limiting absorption [9] must be used).

A certain limitation of the asymptotic method for investigating the problem of internal wave radiation should be noted. Within the framework of the linear mathematical model when using the boundary function method, the properties of the solutions of regular problems, which describe the behaviour of the fluid at points far from the boundary, are determined by the properties of the degenerate operator L_0, which corresponds to an ideal fluid.

Since $\Pi_1\Phi = 0$, the function Φ_0^r is determined from problem (2.13) in which $f_0(\sigma_1, \sigma_2) = 0$. According to formula (2.15) the homogeneous problem (2.13) has only a trivial solution. Consequently, $\Phi_0^r = 0$.

For the regular term Φ_1^r we obtain problem (2.13), where

$$f_1(\sigma_1, \sigma_2) = -\left[\frac{\partial^2 \Pi_2 \Phi}{\partial \sigma_2 \partial \rho} + a(0)\frac{\partial \Pi_1 \Psi}{\partial \sigma_1}\right]_{\rho = 0}$$

and $\Pi_1\Psi$, $\Pi_2\Psi$ are defined by formulae (2.11) and (2.12). Since the function $f_1(\sigma_1, \sigma_2)$ obeys boundary condition (2.14), a unique solution of this problem exists, which can be presented in the form (2.15).

From Eq. (2.6) and boundary conditions (2.8) and (2.10) we find

$$\frac{\partial^2 \Pi_2\Psi}{\partial\rho^2} + i\omega\Pi_2\Psi = -b(0)\frac{\partial \Pi_1\Psi}{\partial\rho}$$

$$\left.\frac{\partial \Pi_2\Psi}{\partial\rho}\right|_{\rho=0} = a(0)\left.\frac{\partial^2\Phi_1^r}{\partial\sigma_2\partial\sigma_1}\right|_\Gamma, \quad \Pi_2\Psi(\rho\to\infty)\to 0$$

and hence

$$\Pi_2\Psi(\rho, \sigma_1, \sigma_2) = \left[\frac{1}{\lambda_1}a(0)\left.\frac{\partial^2\Phi_1^r}{\partial\sigma_2\partial\sigma_1}\right|_\Gamma + \frac{b(0)}{2\lambda_1^3}(\lambda_1\rho - 1)u^1(\sigma_1, \sigma_2)\right]e^{\lambda_1\rho}$$

The regular term of the second approximation satisfies problem (2.13) where

$$f_2(\sigma_1, \sigma_2) = -\left[\frac{\partial^2\Pi_3\Phi}{\partial\sigma_2\partial\rho} + a(0)\frac{\partial\Pi_2\Psi}{\partial\sigma_1}\right]_{\rho=0}$$

For a unique determination of the function Φ_2^r it is necessary to find the boundary function of the third approximation $\Pi_3\Phi$.

The equation for finding $\Pi_3\Phi$ is obtained by equating the coefficients of ε in Eq. (2.5). The corresponding boundary conditions follow from (2.9) and (2.10). We have

$$\begin{gathered}
i\omega\frac{\partial^4}{\partial\rho^4}\Pi_3\Phi - (\omega^2 - N^2)\frac{\partial^2}{\partial\rho^2}\Pi_3\Phi = -b(0)\left[2i\omega\frac{\partial^3}{\partial\rho^3} - (\omega^2 - N^2)\frac{\partial}{\partial\rho}\right]\Pi_2\Phi \\
\left.\frac{\partial^2\Pi_3\Phi}{\partial\rho^2}\right|_{\rho=0} = \\
= \left[-b(0)\left(\frac{\partial\Phi_1^r}{\partial n} + \frac{\partial\Pi_2\Phi}{\partial\rho}\right) - a(0)\left(d_1(0)\frac{\partial\Phi_1^r}{\partial\sigma_1} + a(0)\frac{\partial^2\Phi_1^r}{\partial\sigma_1^2}\right) - \frac{\partial^2\Phi_1^r}{\partial n^2}\right]_\Gamma \equiv g(\sigma_1, \sigma_2) \\
\Pi_3\Phi(\rho\to\infty)\to 0
\end{gathered} \tag{2.17}$$

The solution of problem (2.17) has the form

$$\Pi_3\Phi(\rho) = Ce^{\lambda_2\rho} - \frac{b(0)u^2(\sigma_1, \sigma_2)(2i\omega\lambda_2^2 - (\omega^2 - N^2))}{2\lambda_2^2(\omega^2 - N^2) - 4i\omega\lambda_2^4}\rho e^{\lambda_2\rho}$$

where

$$C = \frac{1}{\lambda_2^2}\left[g(\sigma_1, \sigma_2) + \frac{b(0)u^2(\sigma_1, \sigma_2)(2i\omega\lambda_2^2 - (\omega^2 - N^2))}{\lambda_2(\omega^2 - N^2) - 2i\omega\lambda_2^3}\right]$$

Since the coordinates (r, σ_1, σ_2) were introduce locally, the Π-functions have meaning only in a small vicinity of the boundary Γ. For their smooth extension to the whole region beyond the cylinder, one can use the well-known standard procedure [5].

3. CONCLUSIONS

1. The solution of the problem of forming periodic disturbances constructed above asymptotically exactly satisfies the system of equations and boundary conditions. The sums

$$\varepsilon\Phi_1^r(x, y, z) + \sum_{l=0}^{3} \varepsilon^l \Pi_l \Phi(\rho, \sigma_1, \sigma_2) \text{ and } \sum_{l=0}^{1} \varepsilon^l \Pi_l \Psi(\rho, \sigma_1, \sigma_2)$$

exist and satisfy Eqs (2.1), boundary conditions (2.7) and (2.9) with accuracy $O(\varepsilon^2)$ and boundary condition (2.8) with accuracy $O(\varepsilon)$.

2. In a viscous stratified fluid, unlike the case of a homogeneous one, two boundary layers are formed, which confirms the conclusions reached in [2], obtained by another method. *A viscous wave boundary layer*, which exists both in homogeneous and heterogeneous fluid, is described by the boundary layer expansion $\Pi\Psi(\rho, \sigma_1, \sigma_2, \varepsilon)$ that begins from the term of order $\sqrt{\nu}$. The thickness of the layer is

$$\delta_\nu = \sqrt{2\nu/\omega}$$

The boundary layer correction $\Pi\Phi(\rho, \sigma_1, \sigma_2, \varepsilon)$ describes the *internal wave boundary layer*, which is a specific feature for a stratified fluid. This expansion begins from the term of order ν. The thickness of the layer is

$$l_\nu = \sqrt{\frac{2\omega\nu}{|\omega^2 - N^2|}}$$

On changing to a homogeneous fluid ($N \to 0$) the viscous and internal boundary layers merge: $l_\nu = \delta_\nu = \sqrt{2\nu/\omega}$ and this agrees with the conclusions reached in [2].

3. If we reject the consideration of linearized equations, all the structural elements of the flows begin to interact with each other, and when investigating the dynamics of heterogeneous media the complete non-stationary system (1.1) must be analysed.

4. The method of boundary functions, the use of which is based on extracting small parameters in the system and reducing it to a system of lower dimensionality far from the disturbing surface, enables one to successfully solve both non-stationary and non-linear problems, which this enables one to investigate not only the process by which the wave motion is established, but also new properties of the solutions caused by the non-linearity of the system (contrast structures).

This research was partially supported financially by the Russian Ministry of Education (The Federal Special Purpose "Integration" Programme, No Ya0058/993).

REFERENCES

1. CHERNOUS'KO, F. L., Free oscillations of a viscous fluid in a vessel. *Prikl. Mat. Mekh.*, 1966, **30**, 5, 836–847.
2. KISTOVICH, Yu. V. and CHASHECHKIN, Y. D., Some exactly solvable problems of the radiation of three-dimensional periodic internal waves. *Zh. Prikl. Mekh. Tekh. Fiz.*, 2001, **42**, 2, 52–61.
3. KISTOVICH, Yu. V. and CHASHECHKIN, Yu. D., Non-linear generation of periodic internal waves by a rotating axis-symmetric body. *Dokl. Ross. Akad. Nauk*, 1999, **367**, 5, 636–639.
4. KISTOVISH, Yu. V. and CHASHECHKIN, Yu. D., A new mechanism for the non-linear generation of internal waves. *Dokl. Ross. Akad. Nauk*, 2002, **382**, 6, 772–776.
5. VASIL'YEVA, A. B. and BUTUZOV, V. F., *Asymptotic Methods in the Theory of Singular Perturbations*. Vysshaya Shkola, Moscow, 1990.
6. NEFEDOV, N. N., On some singularly perturbed problem for viscous stratified fluid. *J. Math. Anal. Appl.*, 1988, **131**, 1, 118–126.
7. BUTUZOV, V. F., NEFEDOV, N. N. and FEDOTOVA, Ye. V., Asymptotic solution of the linearized problem of the propagation of sound in a bounded medium with low viscosity. *Zh. Vychisl. Mat. Mat. Fiz.*, 1987, **27**, 2, 226–236.
8. BUTUZOV, B. F., NEFEDOV, N. N. and POLEZHAYEVA, Ye. V., Asymptotic solution of linearized problems on the natural and forced oscillation of a medium with low viscosity. *Zh. Vychisl. Mat. Mat. Fiz.*, 1989, **29**, 7, 1023–1035.
9. VLADIMIROV, V. S., *The Equations of Mathematical Physics*. Nauka, Moscow, 1971.

Translated by E.T.